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p-Adic interpolation and approximation of a continuous function by linear combinations of shifts of p-adic valuations

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Abstract

In this work, questions about interpolation and approximation of a continuous function $f : \mathbb{Z}_p \to \mathbb{R}$ $(f : \mathbb{Z}_p \to \mathbb{Q}_p)$ by functions of the form $\sum_{k=1}^{N} \lambda_k |x - x_k|_p$ are discussed. The theorem about uniform approximation of a continuous function $f : \mathbb{Z}_p \to \mathbb{R}$ is proved. Nonexistence of such approximation for a $\mathbb{Q}_p$-valued function is shown.

1. Introduction

A basic motivation to write this article was to develop some interpolation principle for approximation of a function of p-adic argument. During the last 10 years, p-adic numbers were used intensively in quantum physics, see, for example, books [6,12]. Thus, it is natural to develop an analysis of p-adic functions and approximation theory in such a direction. There were many papers about p-adic interpolation and approximation of a continuous function but most of them were devoted to some generalizations of p-adic interpolation theorems of Mahler and Dieudonne, see

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[3,7,8]. Works such as [1,2,5,9,11] contain such results. We also point out the article of Grintsyavichyus and Markshaĭtis [4] as more related to the topic of our work. The above articles except the last one dealt with interpolation and approximation of a $\mathbb{Q}_p$-valued function. The results of our article refer to approximation for a real-valued continuous function on the p-adic integers. At the basis of the results lies the very simple idea that any real-valued continuous function of real argument can be interpolated and approximated by a piecewise linear continuous function. Here, we propose to interpolate a real-valued continuous function on p-adic integers by p-adic analog of the piecewise linear continuous function. Furthermore, we will prove that the interpolating function is an approximating one in the uniform metric.

Let us define some basic notations. p denotes a prime number. Let $\mathbb{N}$ and $\mathbb{N}_0$ be a set of positive and nonnegative integers, respectively. Denote by $|\cdot|$ or $|\cdot|_\infty$ a standard valuation and by $|\cdot|_p$ a p-adic one. We use the symbol $|\cdot|_{\infty,p}$ when we do not make a difference between $|\cdot|_\infty$ and $|\cdot|_p$, i.e. formulas are true for both valuations. Let $B_\gamma(y) = \{x \in \mathbb{Q}_p \colon |x-y|_p \leqslant p^\gamma\}$. Designations $B_0(0)$, B_0, $\mathbb{Z}_p$ are equivalent. Denote an indicator or a characteristic function of a ball $B_\gamma(y)$ by $I_{B_\gamma(y)}(x)$. For $n \in \mathbb{N}_0$, we set $p^n \times p^n$ block-diagonal matrices $I_k(n) = (\delta_{[\frac{i-1}{p^k}],[\frac{j-1}{p^k}]})_{i,j=1}^{p^n}$, $k = 0, 1, \ldots, n$, where $\delta_{x,y}$ equals one if $x = y$ or zero if $x \neq y$ and $[\cdot]$ denotes an entire part of a number. These matrices have two properties: (1) $\mathrm{Diag}\{\overbrace{I_k(n), \ldots, I_k(n)}^{p}\} = I_k(n+1)$ and (2) $I_i(n) \cdot I_j(n) = I_j(n) \cdot I_i(n) = p^i \cdot I_j(n)$, when $0 \leqslant i \leqslant j \leqslant n$. Let $\{e_i\}_{i=1}^{p^n}$ be a standard basis of vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{p^n}$.

2. Statement and solution of the interpolation problem

Let f be a continuous function on $\mathbb{Z}_p$. For every $n \in \mathbb{N}$ we have uniformly distributed points $1, 2, \ldots, p^n$. Let us consider p-adic expansion of these points and enumerate them by the lexicographical order. After that we shall obtain points $x_1, x_2, \ldots, x_{p^n}$. They are centers of balls $B_{-n}(x_k)$, $k = 1, 2, \ldots, p^n$ and $\mathbb{Z}_p = \coprod_{k=1}^{p^n} B_{-n}(x_k)$. In particular, when $p = 5$ and $n = 2$ the enumeration is shown in Fig. 1. Denote $y_k \coloneqq f(x_k)$, $k = 1, 2, \ldots, p^n$. The problem is to find coefficients λ_k for the function

$$M_n(x) = \sum_{k=1}^{p^n} \lambda_k |x - x_k|_p \tag{1}$$

such that

$$M_n(x_i) = y_i \quad \forall i = 1, 2, \ldots, p^n. \tag{2}$$

We will be able to construct a matrix of system (2) inductively.

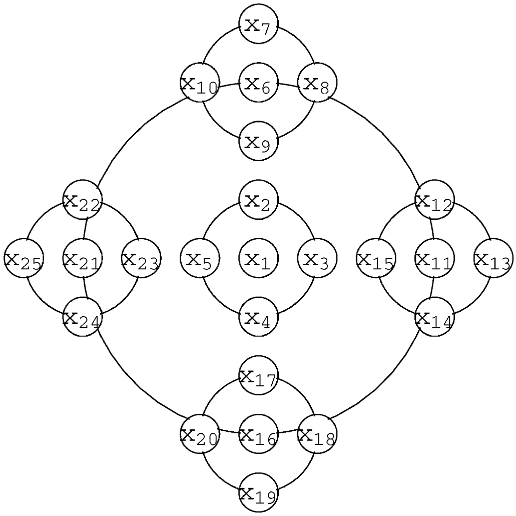

Fig. 1. The enumeration of the partition of Z_5.

Denote by $K_n = (|x_i - x_k|_p)_{i,k=1}^{p^n}$, $n = 1, 2, \ldots$ the matrix of (2).

Lemma 1. *The following expansion of* K_n *is valid*:

$$K_n = -\frac{1}{p^{n-1}} I_0(n) - \sum_{k=1}^{n-1} \left(\frac{p-1}{p^{n-k}} I_k(n) \right) + I_n(n). \tag{3}$$

Proof. Here, we use a self-similarity of $\mathbb{Z}_p$ and the first property of $I_k(n)$ $0 \leqslant k \leqslant n$ (see Introduction).

For $n = 1$ we have

$$K_1 = -I_0(1) + I_1(1).$$

For $n = 2$,

$$\begin{aligned} K_2 &= I_2(2) - I_1(2) + p^{-1} \operatorname{Diag}\{\overbrace{K_1, \ldots, K_1}^{p}\} \\ &= I_2(2) - I_1(2) + p^{-1}(-I_0(2) + I_1(2)) \\ &= -p^{-1} I_0(2) - \frac{p-1}{p} I_1(2) + I_2(2). \end{aligned}$$

When $n=3$ we obtain that

$$K_3 = I_3(3) - I_2(3) + p^{-1}\,\mathrm{Diag}\{\overbrace{K_2, \ldots, K_2}^{p}\}$$
$$= I_3(3) - I_2(3) + p^{-1}\left(-p^{-1}I_0(3) - \frac{p-1}{p}I_1(3) + I_2(3)\right)$$
$$= -\frac{1}{p^2}I_0(3) - \frac{p-1}{p^2}I_1(3) - \frac{p-1}{p}I_2(3) + I_3(3).$$

Suppose that (3) is true for K_{n-1} (inductive assumption). When the subscript of the matrix of (2) equals n we get

$$K_n = I_n(n) - I_{n-1}(n) + p^{-1}\,\mathrm{Diag}\{\overbrace{K_{n-1}, \ldots, K_{n-1}}^{p}\}$$
$$= I_n(n) - I_{n-1}(n) + p^{-1}\left(-\frac{1}{p^{n-2}}I_0(n) - \sum_{k=1}^{n-2}\frac{p-1}{p^{n-k-1}}I_k(n) + I_{n-1}(n)\right),$$

which equals (3). □

Now, to solve (2) we should invert K_n. But first we prove the next lemma. Further, we will omit (n) in formulas with $I_k(n)$.

Lemma 2 (Criterion of invertibility of a matrix from span$\langle I_0, I_1, \ldots, I_n\rangle$). *A matrix* $A = \sum_{k=0}^{n} a_k I_k$ *is an invertible iff* $\sum_{i=0}^{k} a_i p^i \neq 0$ *for all* $k = 0, 1, \ldots, n$ *and* $A^{-1} = \sum_{k=0}^{n} b_k I_k$, *where* $b_0 = \frac{1}{a_0}$,

$$b_k = -\frac{a_k}{\sum_{i=0}^{k-1} a_i p^i \sum_{i=0}^{k} a_i p^i}, \quad k = 1, 2, \ldots, n. \tag{4}$$

Proof. Recall that matrices $\{I_k\}_{k=0}^{n}$ have the following property:

$$I_i \cdot I_j = I_j \cdot I_i = p^i I_j, \quad \text{if } 0 \leqslant i \leqslant j \leqslant n. \tag{5}$$

It means that span$\langle I_0, I_1, \ldots, I_n\rangle$ is an algebra. Therefore, if A^{-1} exists, then, it must have the same expansion as A. Namely, $A^{-1} = \sum_{j=0}^{n} b_j I_j$, where $b_j,\ j = 0, 1, \ldots, n$ are unknown coefficients which can be derived from the identity

$$\sum_{k=0}^{n} a_k I_k \cdot \sum_{j=0}^{n} b_j I_j = I_0.$$

Let us multiply the left-hand side of the last equation taking into account (5) and equate coefficients with I_k on both sides. We have

$$a_0 \cdot b_0 = 1,$$
$$a_k \sum_{i=0}^{k-1} p^i b_i + \left(\sum_{i=0}^{k} a_i p^i\right) b_k = 0, \quad k = 1, 2, \ldots, n.$$

This is the system of linear equations with respect to $b_k,\ k = 0, 1, \dots, n$ which has a solution iff $\sum_{i=0}^{k} a_i p^i \neq 0,\ k = 0, 1, \dots, n$ and next recurrent relations hold:

$$b_0 = \frac{1}{a_0},$$
$$b_k = \frac{-a_k(\sum_{i=0}^{k-1} b_i p^i)}{\sum_{i=0}^{k} a_i p^i}, \quad k = 1, 2, \dots, n. \tag{6}$$

Remark that if $a_k = 0$ for some indexes $k \in \{1, \dots, n\}$, then $b_k = 0$ and formula (4) is true. So we eliminate from considerations all $a_k = 0$. Let $a_{k_l} \neq 0,\ k_l \in \Lambda \subset \{0, 1, 2, \dots, n\}$ for all $l = 0, 1, 2, \dots, L$. We set $a_{k_0} := a_0,\ b_{k_0} := b_0$. Therefore, we have the following relations:

$$b_{k_0} = \frac{1}{a_{k_0}},$$
$$b_{k_l} = \frac{-a_{k_l}(\sum_{i=0}^{k_{l-1}} b_i p^i)}{\sum_{i=0}^{k_l} a_i p^i}, \quad l = 1, 2, \dots, L. \tag{7}$$

We can rewrite these formulas as

$$b_{k_0} = \frac{1}{a_{k_0}},$$
$$b_{k_1} = \frac{-a_{k_1} \cdot b_{k_0}}{\sum_{i=0}^{k_1} a_i p^i},$$
$$b_{k_l} = \frac{a_{k_l}}{a_{k_{l-1}}} \frac{\sum_{i=0}^{k_{l-2}} a_i p^i}{\sum_{i=0}^{k_l} a_i p^i} b_{k_{l-1}}, \quad l = 2, 3, \dots, L. \tag{8}$$

Let us multiply $j + 1$ first equations in (8) and reduce coefficients $b_{k_0}, b_{k_1}, \dots, b_{k_{j-1}}$ from both sides. We have

$$\begin{aligned} b_{k_j} &= -\frac{1}{a_{k_0}} \frac{a_{k_1}}{\sum_{i=0}^{k_1} a_i p^i} \prod_{l=2}^{j} \left(\frac{a_{k_l}}{a_{k_{l-1}}} \frac{\sum_{i=0}^{k_{l-2}} a_i p^i}{\sum_{i=0}^{k_l} a_i p^i} \right) \\ &= -\frac{a_{k_j}}{\sum_{i=0}^{k_{l-1}} a_i p^i \sum_{i=0}^{k_l} a_i p^i}, \quad j = 2, 3, \dots, L. \end{aligned} \tag{9}$$

From (8) one can conclude that the last equality in (9) holds for $j = 1$ too. Since $k_{l-1} \leqslant k_l - 1$, then $a_i = 0$ for all $i \in (k_{l-1}, k_l - 1)$. Therefore, $\sum_{i=0}^{k_{l-1}} a_i p^i = \sum_{i=0}^{k_l - 1} a_i p^i$. Finally, we can rewrite (9) as

$$b_{k_0} = \frac{1}{a_{k_0}},$$
$$b_{k_j} = -\frac{a_{k_j}}{\sum_{i=0}^{k_l - 1} a_i p^i \sum_{i=0}^{k_l} a_i p^i}, \quad j = 1, 2, \dots, L. \tag{10}$$

Comparing (10) and (4) we see that both cases $a_k = 0$ and $a_k \neq 0$ can be described by (4). □

Theorem 1. *Let f be a function acting from $\mathbb{Z}_p$ to $\mathbb{R}$ or $\mathbb{Q}_p$. Let $\{x_k\}_{k=1}^{p^n}$ be a sequence $1, 2, \ldots, p^n$ ordered by the lexicographical law. Then there exists the unique function $M_n(x) = \sum_{k=1}^{p^n} \lambda_k |x - x_k|_p$ such that $M_n(x_i) = f(x_i)$ for all $i = 1, 2, \ldots, p^n$.*

Proof. Applying Lemma 2 to K_n from (3) we find $K_n^{-1} = \sum_{k=0}^{n} b_k I_k$, where

$$b_0 = -p^{n-1},$$
$$b_k = \frac{(p+1)(p^2-1)}{(p^{2k-1}+1)(p^{2k+1}+1)} \cdot p^{n+k-2}, \quad 1 \leqslant k \leqslant n-1,$$
$$b_n = \frac{(p+1)^2}{(p^{2n}-1)(p^{2n-1}+1)} \cdot p^{2n-2}. \tag{11}$$

Now we have to find $\lambda_i,\ i = 1, \ldots, p^n$. The above considerations imply that

$$\sum_{i=1}^{p^n} \lambda_i e_i = K_n^{-1} \cdot \sum_{i=1}^{p^n} y_i e_i = \sum_{k=0}^{n} b_k I_k \sum_{i=1}^{p^n} y_i e_i = \sum_{k=0}^{n} \sum_{i=1}^{p^n} y_i b_k I_k e_i. \tag{12}$$

The last formula in (12) contains the expression $I_k e_i$. It is not hard to see that $I_0 e_i = e_i$ and $I_k e_i = \sum_{j=1+[\frac{i-1}{p^k}]p^k}^{(1+[\frac{i-1}{p^k}])p^k} e_j$ for all $k = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, p^n$. Substituting the value of $I_k e_i$ into (12) we get

$$\sum_{i=1}^{p^n} \lambda_i e_i = b_0 \sum_{i=1}^{p^n} y_i e_i + \sum_{k=1}^{n} b_k \sum_{i=1}^{p^n} y_i \sum_{j=1+[\frac{i-1}{p^k}]p^k}^{(1+[\frac{i-1}{p^k}])p^k} e_j \tag{13}$$

(here and below $[\cdot]$ denotes an entire part of a number). Since the conditions

$$1 \leqslant i \leqslant p^n \quad \text{and} \quad 1 + \left[\frac{i-1}{p^k}\right] p^k \leqslant j \leqslant \left(1 + \left[\frac{i-1}{p^k}\right]\right) p^k$$

are equivalent with

$$1 \leqslant i \leqslant p^n \quad \text{and} \quad \left[\frac{i-1}{p^k}\right] = \left[\frac{j-1}{p^k}\right],$$

we obtain

$$\sum_{i=1}^{p^n} \lambda_i e_i = b_0 \sum_{i=1}^{p^n} y_i e_i + \sum_{k=1}^{n} b_k \sum_{i=1}^{p^n} e_i \sum_{j=1+[\frac{i-1}{p^k}]p^k}^{(1+[\frac{i-1}{p^k}])p^k} y_j, \tag{14}$$

which yields

$$\lambda_i = b_0 y_i + \sum_{k=1}^{n} b_k \sum_{j=1+[\frac{i-1}{p^k}]p^k}^{(1+[\frac{i-1}{p^k}])p^k} y_j, \quad i = 1, 2, \dots, p^n. \tag{15}$$

Since (2) has the unique solution (15), the function M_n exists and is unique. □

Remark 1. The lexicographical ordering of $1, 2, \dots, p^n$ was necessary just to invert the matrix of distances K_n. Since the value of sum (1) does not depend on an order of summation, we may write that $M_n(x) = \sum_{i=1}^{p^n} \lambda_{k_i} |x - i|_p$, where $(k_i)_{i=1}^{p^n}$ is a corresponding permutation of $(1, 2, \dots, p^n)$.

3. Statement and solution of the approximation problem

Proceeding from the previous section we have the questions. Does the function M_n converge to f uniformly when n tends to infinity? What restrictions on function f imply such convergence? Answers will be given in Theorems 2 and 3. But first we have to prove the lemma.

Lemma 3. *The following formula holds*:

$$\max_{y \in B_0} \left| M_n(y) - \sum_{i=1}^{p^n} y_i I_{B_{-n}(x_i)}(y) \right|_\infty = \max_{l=1,\dots,p^n} |\lambda_l|_\infty \cdot \frac{1}{p^n}. \tag{16}$$

Proof. Let us find the difference between M_n and

$$f_n = \sum_{i=1}^{p^n} y_i I_{B_{-n}(x_i)}. \tag{17}$$

By the construction of M_n we have that $M_n(x_i) = y_i,\ i = 1, 2, \dots, p^n$. Let y be an arbitrary point in B_0 such that $y \neq x_k\ \forall k = 1, 2, \dots, p^n$. Since $B_0 = \coprod_{k=1}^{p^n} B_{-n}(x_k)$, there is a ball $B_{-n}(x_i)$ such that $y \in B_{-n}(x_i) \setminus \{x_i\}$. As a result of this we can write

$$M_n(y) = \sum_{k=1}^{i-1} \lambda_k |y - x_k|_p + \lambda_i |y - x_i|_p + \sum_{k=i+1}^{p^n} \lambda_k |y - x_k|_p.$$

Since a distance between points of different balls is equal to the distance between the centers, we can replace y in the first and last sums with x_i. After that, we get the equality $M_n(y) = M_n(x_i) + \lambda_i |y - x_i|_p,\ y \in B_{-n}(x_i)$ which can be performed to $M_n(y) = y_i + \lambda_i |y - x_i|_p,\ y \in B_{-n}(x_i)$. If $y \in B_{-n}(x_j),\ j \neq i$ then $M_n(y) = y_j + \lambda_j$

$|y - x_j|_p$. Resuming last two equalities one can conclude that

$$M_n(y) = \sum_{i=1}^{p^n} y_i I_{B_{-n}(x_i)}(y) + \sum_{i=1}^{p^n} \lambda_i |y - x_i|_p I_{B_{-n}(x_i)}(y).$$

This obviously gives us

$$\sup_{y \in B_0} \left| M_n(y) - \sum_{i=1}^{p^n} y_i I_{B_{-n}(x_i)}(y) \right|_{\infty,p} = \sup_{y \in B_0} \left| \sum_{i=1}^{p^n} \lambda_i |y - x_i|_p I_{B_{-n}(x_i)}(y) \right|_{\infty,p}.$$

In the case of a standard norm $|\cdot|_\infty$, the right-hand side of the equality is a maximum of a continuous function over the compact set B_0. Therefore, it is attained at the point $\tilde{y}$. So we can write

$$\sup_{y \in B_0} \left| M_n(y) - \sum_{i=1}^{p^n} y_i I_{B_{-n}(x_i)}(y) \right|_{\infty} = |\lambda_l|_\infty \cdot |\tilde{y} - x_l|_p, \quad \tilde{y} \in B_{-n}(x_l).$$

In this case, the point $\tilde{y}$ must belong to $B_{-n}(x_l) \setminus B_{-n-1}(x_l)$ and $|\lambda_l|$ must be maximal. Otherwise, we will get a contradiction with the maximality of $|\sum_{i=1}^{p^n} \lambda_i |y - x_i|_p I_{B_{-n}(x_i)}(y)|_\infty$. Therefore, we obtain formula (16). □

Theorem 2. *Let $f : \mathbb{Z}_p \to \mathbb{R}$ be a continuous function. Then for any $\varepsilon > 0$ there exists a positive integer N_ε such that for all $n > N_\varepsilon$*

$$|M_n(x) - f(x)| < \varepsilon, \quad \forall x \in \mathbb{Z}_p. \tag{18}$$

Proof. Let us split our proof into two steps. The first step is to prove the theorem for a characteristic function of a ball. The second step is to prove the theorem for an arbitrary continuous function.

Step 1: We shall approximate a characteristic function $I_{B_{-m}(x_l)}$, where $B_{-m}(x_l)$ is an arbitrary ball in $\mathbb{Z}_p$. Without loss of generality, we can assume that $x_l \in \mathbb{N}_0$. In a contrary case, there is a point $x_l^* \in \mathbb{N}_0$ (since $\mathbb{N}_0$ is dense in $\mathbb{Z}_p$) such that $|x_l - x_l^*|_p < p^{-m}$ and the equality $I_{B_{-m}(x_l)} = I_{B_{-m}(x_l^*)}$ is true. Therefore, one can assume that x_l is a center of $B_{-m}(x_l)$ and x_l is a point from the partition $x_1, x_2, \ldots, x_{p^m}$, $\mathbb{Z}_p = \coprod_{i=1}^{p^m} B_{-m}(x_i)$. Let us divide every ball $B_{-m}(x_i)$ into p^n balls. After that we obtain the partition $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{p^{n+m}}$, where $\hat{x}_{p^n(l-1)+1}, \ldots, \hat{x}_{p^n l} \in B_{-m}(x_l)$. Then $y_i = 1$ if $i = p^n(l-1)+1, \ldots, p^n l$ and $y_i = 0$ otherwise. We can construct the function $M_{n+m}(x) = \sum_{i=1}^{p^{n+m}} \lambda_i |x - \hat{x}_i|_p$, where λ_i can be found from the formula

$$\lambda_i = b_0 y_i + \sum_{k=1}^{n+m} b_k \sum_{j=1+[\frac{i-1}{p^k}]p^k}^{(1+[\frac{i-1}{p^k}])p^k} y_j, \quad i = 1, 2, \ldots, p^{n+m}. \tag{19}$$

Taking into account the values of y_j we can rewrite (19) as

$$\lambda_i = (b_n + b_{n+1} + \cdots + b_{n+m})p^n \quad \text{for } 1 \leqslant i \leqslant p^n(l-1), \tag{20}$$

$$\lambda_i = \sum_{k=0}^{n} b_k p^k + \sum_{k=n+1}^{n+m} b_k p^n \quad \text{for } p^n(l-1)+1 \leqslant i \leqslant p^n l, \tag{21}$$

$$\lambda_i = (b_{n+1} + \cdots + b_{n+m})p^n \quad \text{for } p^n l + 1 \leqslant i \leqslant p^{n+m}, \tag{22}$$

where

$$\begin{aligned} b_0 &= -p^{n+m-1}, \\ b_k &= \frac{p^{k+n+m-2}(p+1)(p^2-1)}{(p^{2k-1}+1)(p^{2k+1}+1)} \quad \text{for } k = 1, 2, \ldots, n+m-1, \\ b_{n+m} &= \frac{p^{2n+2m-2}(p+1)^2}{(p^{2n+2m-1}+1)(p^{2n+2m}-1)}. \end{aligned} \tag{23}$$

Remark that if $m = 0$, then $l := 1$ and we will obtain only the first sum in (21). Denote by $B_n := p^n \sum_{k=n+1}^{n+m} b_k$. Then (20)–(22) are rewritten as

$$\lambda_i = b_n p^n + B_n \quad \text{for } 1 \leqslant i \leqslant p^n(l-1), \tag{24}$$

$$\lambda_i = \sum_{k=0}^{n} b_k p^k + B_n \quad \text{for } p^n(l-1)+1 \leqslant i \leqslant p^n l, \tag{25}$$

$$\lambda_i = B_n \quad \text{for } p^n l + 1 \leqslant i \leqslant p^{n+m}. \tag{26}$$

Let us substitute b_k from (23) into (24)–(26) and find an asymptotic for $|\lambda_i|$. Using (23)

$$p^\gamma < (p^\gamma + 1) < 2p^\gamma \quad \text{and} \quad \tfrac{1}{2}p^\gamma \leqslant p^\gamma - 1 < p^\gamma, \quad \gamma, m \in \mathbb{N}, \tag{27}$$

it is easy to see that

$$\tfrac{1}{4}(p+1)(p^2-1)p^{-3k+n+m-2} < b_k < (p+1)(p^2-1)p^{-3k+n+m-2},$$

when $k = n+1, \ldots, n+m-1$

$$\tfrac{1}{2}(p+1)^2 p^{-2n-m-1} \leqslant b_{n+m} < 2(p+1)^2 p^{-2n-m-1}.$$

By notation of B_n we obtain that

$$\begin{aligned} &\frac{1}{4}(p+1)(p^2-1)p^{2n+m-2} \sum_{k=n+1}^{n+m-1} p^{-3k} + \frac{1}{2}(p+1)^2 p^{-2n-m-1} \\ &\quad < B_n < (p+1)(p^2-1)p^{2n+m-2} \sum_{k=n+1}^{n+m-1} p^{-3k} + (p+1)^2 p^{-2n-m-1}. \end{aligned}$$

The sum in the last formula can be reduced as a sum of a finite geometric progression. Therefore, we have

$$\begin{aligned} &\tfrac{1}{4}C_1(p+1)(p^2-1)p^{-n+m-5} + \tfrac{1}{2}(p+1)^2 p^{-n-3m-1} < B_n \\ &\quad < C_1(p+1)(p^2-1)p^{-n+m-5} + 2(p+1)^2 p^{-n-3m-1}, \end{aligned} \tag{28}$$

where $C_1 = \frac{1-p^{-3m-3}}{1-p^{-3}}$. That means

$$B_n = O(p^{-n+m-2}), \quad n \to \infty. \tag{29}$$

Since (see in (23))

$$b_n p^n = \frac{p^{3n+m-2}(p+1)(p^2-1)}{(p^{2n-1}+1)(p^{2n+1}+1)}$$

and using (27) one can conclude that

$$\tfrac{1}{4}(p+1)(p^2-1)p^{-2n+m-2} < b_n p^n < (p+1)(p^2-1)p^{-2n+m-2}.$$

So, it has an asymptotic equality

$$b_n p^n = O(p^{-2n+m+1}), \quad n \to \infty,$$

which together with (29) gives us a formula

$$b_n p^n + B_n = O(p^{-n+m-2}), \quad n \to \infty. \tag{30}$$

Comparing (30) with (24) and (29) with (26), we can say that for $|\lambda_i|$ from (24) and (26) the following asymptotic formula is valid:

$$|\lambda_i| = O(p^{-n+m-2}), \quad n \to \infty, \tag{31}$$

where

$$i \in \{1, \dots, p^n(l-1)\} \cup \{p^n l + 1, \dots, p^{n+m}\}.$$

Let us find an asymptotic for λ_i from (25). Using

$$\frac{(p^2-1)p^{2k-1}}{(p^{2k-1}+1)(p^{2k+1}+1)} = \frac{1}{p^{2k-1}+1} - \frac{1}{p^{2k+1}+1}$$

we obtain that

$$\sum_{k=0}^{n} b_k p^k = -p^{n+m-1}\frac{p+1}{p^{2n+1}+1}. \tag{32}$$

It yields that

$$\left|\sum_{k=0}^{n} b_k p^k\right| = O(p^{-n+m-1}). \tag{33}$$

Eqs. (29) and (33) imply that

$$|\lambda_i| = O(p^{-n+m-1}), \quad n \to \infty, \; i \in \{p^n(l-1)+1, \dots, p^n l\}. \tag{34}$$

By (31) and (34) we have the asymptotic

$$\max_{i=1,\dots,p^{n+m}} |\lambda_i| = O(p^{-n+m-1}) \quad \text{as } n \to \infty,$$

and by (16) we have that

$$\max_{x\in\mathbb{Z}_p} |M_{n+m}(x) - I_{B_{-m}(x_l)}(x)| = O(p^{-2n-1}), \quad n \to \infty.$$

In other words, there is a positive constant C such that

$$\max_{x\in\mathbb{Z}_p} |M_{n+m}(x) - I_{B_{-m}(x_l)}(x)| \leqslant Cp^{-2n-1} \quad \text{for all } n \geqslant n_0. \tag{35}$$

Step 2: Since any continuous function $f : \mathbb{Z}_p \to \mathbb{R}$ can be uniformly approximated by linear combinations of indicators $I_{B_{-m}(x_l)}$ (see [10, Theorem 26.2]), for any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$\left| f(x) - \sum_{l=1}^{p^m} y_l I_{B_{-m}(x_l)}(x) \right| < \varepsilon \tag{36}$$

for all $x \in \mathbb{Z}_p$. Using the ordinary triangle inequality we can obtain the following estimation:

$$\left| f(x) - \sum_{l=1}^{p^m} y_l M_{n+m}^{(l)}(x) \right| \leqslant \left| f(x) - \sum_{l=1}^{p^m} y_l I_{B_{-m}(x_l)}(x) \right| + \left| \sum_{l=1}^{p^m} y_l I_{B_{-m}(x_l)}(x) - \sum_{l=1}^{p^m} y_l M_{n+m}^{(l)}(x) \right|. \tag{37}$$

Expressions (36) and (37) imply

$$\left| f(x) - \sum_{l=1}^{p^m} y_l M_{n+m}^{(l)}(x) \right| \leqslant \varepsilon + \sum_{l=1}^{p^m} |y_l| |I_{B_{-m}(x_l)}(x) - M_{n+m}^{(l)}(x)|. \tag{38}$$

Applying (35)–(38) and using $|y_l| \leqslant \max_{x\in\mathbb{Z}_p} |f(x)|$ we obtain

$$\left| f(x) - \sum_{l=1}^{p^m} y_l M_{n+m}^{(l)}(x) \right| \leqslant \varepsilon + C \max_{x\in\mathbb{Z}_p} |f(x)| p^m p^{-2n-1}. \tag{39}$$

When n tends to infinity $Cp^m p^{2n-1}$ tends to zero. Therefore, there exists a positive integer N_ε such that for all $n > N_\varepsilon$, $C \max_{x\in\mathbb{Z}_p} |f(x)| p^m p^{2n-1} < \varepsilon$. Therefore, (39) yields $|f(x) - \sum_{l=1}^{p^m} y_l M_{n+m}^{(l)}(x)| \leqslant 2\varepsilon$, for all $n > N_\varepsilon$ and $x \in \mathbb{Z}_p$. Setting $M_{n+m}(x) := \sum_{l=1}^{p^m} y_l M_{n+m}^{(l)}(x)$, we will get the assertion of the theorem. □

Let us remark that the p-adic absolute value of a p-adic integer is an integer power of p, hence, an element of $\mathbb{Q}_p$. That means system (2) can be solved for $y_i \in \mathbb{Q}_p$, $i = 1, 2, \dots, p^n$. Therefore, $\lambda_i \in \mathbb{Q}_p$, $i = 1, 2, \dots, p^n$ and one can consider M_n as a $\mathbb{Q}_p$-valued function. Let us demonstrate what is going on in such a case.

Theorem 3. *Let $f:\mathbb{Z}_p\to\mathbb{Q}_p$ be a continuous function. Then*

$$\sup_{y\in\mathbb{Z}_p} |M_n(y)-f(y)|_p=+\infty. \tag{40}$$

Proof. In the case of p-adic valuation, we cannot use the theorem about the maximum of a continuous function because $|x-x_i|_p$ has a singularity at x_i as a $\mathbb{Q}_p$-valued function. As a supremum of a function always bigger than the value of a function at the point, we may write

$$\sup_{y\in B_0}\left|\sum_{i=1}^{p^n}\lambda_i|y-x_i|_p I_{B_{-n}(x_i)}(y)\right|_p \geqslant |\lambda_l|_p||\hat{y}-x_l|_p|_p,$$

where $\hat{y}$ is an arbitrary point in $B_{-n}(x_l)\setminus\{x_l\}$. If $\hat{y}$ is taken close to x_l, then $|\lambda_l|_p\cdot||\hat{y}-x_l|_p|_p$ tends to infinity. That means (40). □

Remark 2. Theorem 3 shows that $f:\mathbb{Z}_p\to\mathbb{Q}_p$ cannot be approximated. This happens because small numbers $\frac{1}{p^k}$ in $\mathbb{R}$ are large numbers in $\mathbb{Q}_p$. In this case it might be more natural to consider the function of the type

$$M_n(x)=\sum_{k=1}^{p^n}\frac{\lambda_k(1-\delta_{x_k}(x))}{|x-x_k|_p},$$

where $\delta_{x_k}(x)=1$, if $x=x_k$ and $\delta_{x_k}(x)=0$ otherwise.

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